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Letter to the Editor

Free vibration of stepped cantilever Mindlin plates

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1. Introduction

A recent study by Gorman and Singhal [1] outlined the vibration analysis of stepped cantilever plates using a superposition method. Two- and three-step plates were discussed and limited tabular results were given. The current study was partially motivated as a result of studying the work of Gorman and Singhal [1]. The cantilevered plate has been studied by several researchers and there are sufficient analytical results in the literature to verify a numerical analysis of plate vibration. In this report the finite element method of analysis is used to extend the study of cantilevered stepped plates to include moderately thick plates. First order shear deformation effects are included using the Mindlin plate formulation. Concepts that are required for Mindlin plate theory have been given by Reismann [2] and the development of the corresponding finite element concepts has been sufficiently discussed by Reddy [3] and will not be restated in this note.

Vibration results for constant thickness cantilevered plates using Mindlin plate theory have been given by Liew et al. [4]. The finite element results reported here can be verified by comparison with Gorman and Singhal [1], Liew et al. [4] and Gorman [5]. Hull and Buchanan [6] have given a brief history of contributions to stepped plate theory and vibration results for moderately thick plates, but that work focused on simply supported and clamped square plates. An application for stepped cantilevered plate analysis has been discussed by Li [7–9]. Li [7] used a cantilevered beam analogy to study the behavior of shear-wall type buildings. Vibration of non-uniform cantilevered plates is studied [8,9] with application to the analysis of shear walls.

Additionally, it would appear that there are no results in the literature for stepped cantilevered plates using a first order shear deformation theory. In this letter, a finite element based on Mindlin plate theory including the first order shear deformation, MIN6, developed by Liu and Riggs [10,11], will be used for the vibration analysis of stepped cantilevered plates. However, before discussing results for stepped Mindlin plates, the agreement of the finite element analysis with existing cantilevered plate (thin and moderately thick) vibration studies will be verified and documented. Following that, new numerical results of frequencies and mode shapes for stepped Mindlin plates using the MIN6 element will be tabulated and presented graphically.

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2. Numerical model

MIN6 is a higher order, six-node triangular, anisoparametric Mindlin plate element with cubic variation for transverse displacement and quadratic variation for rotational displacements. More details of the derivation for this element can be found in Refs. [10,11]. Because of its excellent performances, that is neither shear locking nor excessive stiffness in the thin limit, the MIN6 element was used to model isotropic thin and moderately thick stepped cantilevered plates for free vibration analysis in this paper. Vibration results using MIN6 for span-thickness ratio $L/t = 1000$ (thin plate) to $L/t = 5$ (moderately thick plate) were compared with results from Gormann [6] (based on thin plate theory) and Liew (based on Mindlin plate theory) as well. MIN6 gave satisfactory results for both thin and moderately thick plates.

3. Validation and numerical results

The frequencies given by Liew et al. [4] for a square cantilevered plate using Mindlin plate theory compare favorably with Table 4.1 of Gorman [5] where frequencies are based upon classical plate theory. The frequency of Ref. [4] is computed using $v = 0.3$ and non-dimensional plate thickness of 0.001 where the frequency is non-dimensional with respect to the plate thickness. Gorman [5] used $v = 0.333$ and mass per unit area, which is equivalent to using a unit non-dimensional plate thickness. It follows that the frequency of Ref. [5] multiplied by π^2 gives 3.474, which compares with 3.459 of Ref. [4], indicating good agreement (less than 0.5%) even though there is a small difference in the Poisson ratio for these results. The solution using the element of this paper is 3.463 with $v = 0.333$.

Two verification tests for cantilevered uniform thickness plates for the MIN6 element have been performed. First, the non-dimensional frequency $\lambda = (\omega L^2/\pi^2)(\rho t/D)^{1/2}$, (where ω is the natural frequency and ρ is the mass per unit volume) of thin ($L/t = 1000$) and moderately thick ($L/t = 20$ and 10) square cantilever Mindlin isotropic plates are obtained and compared with Liew's results

Table 1
Comparison between MIN6 and Liew et al. [4] results for non-dimensional frequency $\lambda = (\omega L^2/\pi^2)(\rho t/D)^{1/2}$ for a square cantilever Mindlin isotropic plate with different span-thickness ratios

Mode number	$L/t = 1000$		$L/t = 20$		$L/t = 10$	
	MIN6	Liew et al. [4]	MIN6	Liew et al. [4]	MIN6	Liew et al. [4]
1	0.352	0.352	0.351	0.350	0.348	0.348
2	0.862	0.862	0.847	0.844	0.819	0.816
3	2.158	2.157	2.123	2.121	2.037	2.034
4	2.758	2.756	2.704	2.698	2.587	2.582
5	3.140	3.136	3.051	3.039	2.870	2.860
6	5.503	5.490	5.274	5.246	4.833	4.811
7	6.220	6.206	5.999	5.989	5.489	5.477
8	6.513	6.499	6.287	6.270	5.788	5.772

Table 2
Comparison between MIN6 and Gorman [5] non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a thin cantilever plate with aspect ratio 3 and uniform thickness

Mode number	$L/t = 1000$	
	MIN6	Gorman [5]
1	31.575	31.473
2	39.472	39.312
3	61.064	60.642
4	97.688	96.57
5	153.19	150.57
6	199.47	197.37
7	207.08	204.21
8	234.34	228.87
9	243.37	237.15
10	297.71	288.90

[4] in Table 1. The Poisson ratio is 0.3. The agreement between the frequencies computed and Liew's frequencies is within 0.5% or less. Secondly, the comparison shown in Table 2 is the non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ with $v = 0.333$ between MIN6 and Table 4.1 of Gormann [5] for a thin ($L/t = 1000$) cantilever plate of aspect ratio 3 with uniform thickness (Fig. 1). The results of Table 2 show good agreement.

Additionally, to validate the accuracy of the MIN6 results for stepped cantilevered plates, the non-dimensional frequencies ($\lambda = \omega L^2(\rho t/D)^{1/2}$) are determined for two cases with the same span-thickness ratio $L/t = 24$ for the thickest segment. One is three segments with aspect ratio 1 of two-step discontinuities in thickness shown in Fig. 2 and the other one is four segments with aspect ratio 3/4 and three-step discontinuities in thickness shown in Fig. 3. The solutions are compared to the Tables 1 and 2 of Gorman and Singhal [1] and are given in Tables 3 and 4. The results demonstrate a very close agreement for both cases. In Table 5, the comparison of the first 6 frequencies (Hz) between MIN6 and Table 3 of Gorman and Singhal [1] is provided. The comparison shows that the computed frequencies and Gorman and Singhal's experimental measured frequencies are in very close agreement. Table 5 also shows that all numerical frequencies of MIN6 are lower than those measured experimentally while Gorman and Singhal's are higher. The plate was assumed to be 16×12 in with thickness of 0.5, 0.375, 0.25, and 0.125 in thickness. Fig. 14 of Ref. [1] indicates that actual dimensions might be slightly different. Young's modulus was used as $E = 10^7$ psi with $v = 0.33$.

New results using the MIN6 element for cantilever plates with different aspect ratios and composed of different segments of equal geometry but of different thickness are given in Tables 6–9. The non-dimensional frequency $\lambda = \omega L^2/(\rho t/D)^{1/2}$ has been computed for the plates shown in Figs. 1–4. The Poisson ratio is 0.333. The first sixteen free vibration modes are tabulated in Tables 6–9 with different L/t ratios ($L/t = 30$, 24, 10, 5) for four different stepped plates. In every case the L/t ratio corresponds to the thickest segment.

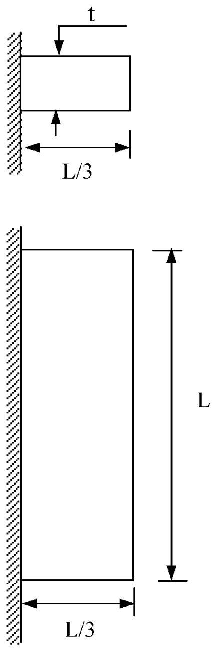

Fig. 1. Cantilever plate with aspect ratio 3.

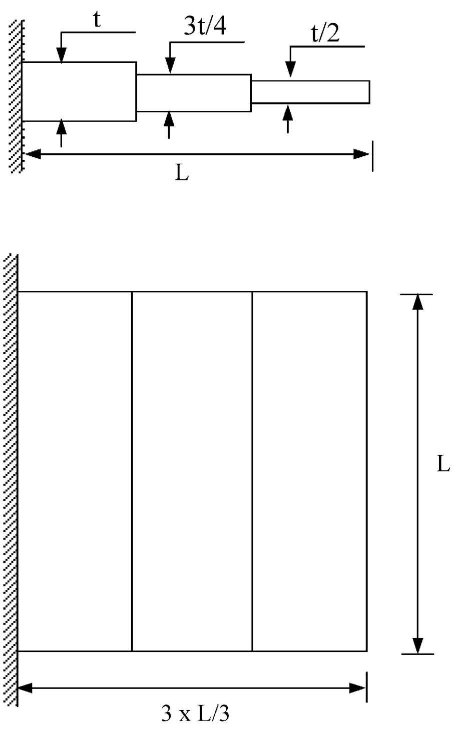

Fig. 2. Square cantilever plate of three segments of equal width with different thickness.

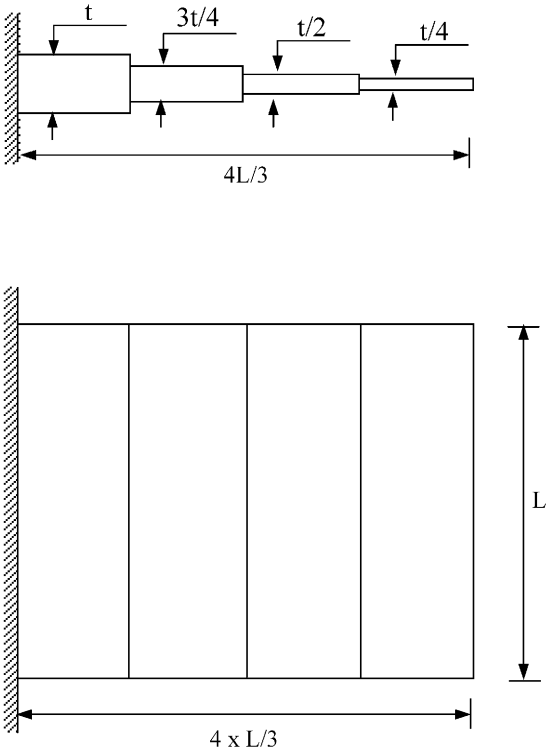

Fig. 3. Cantilever plate with aspect ratio 3/4 of four segments of equal width with different thickness.

Table 3
Comparison between MIN6 and Gorman and Singhal [1] non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a square cantilever Mindlin isotropic plate with two-step discontinuities in thickness

Mode number	Type	$L/t = 24$	
		MIN6	Gorman and Singhal [1]
1	Sym	4.129	4.132
2	Antisym	7.545	7.597
3	Sym	16.43	16.51
4	Antisym	18.64	18.76

Computed mode shapes for the plate with three discontinuities in thickness are shown in Fig. 5 and correspond to $L/t = 24$. Liew et al. [4] show the first eight mode shapes for a square plate with $L/t = 0.1$. The first three mode shapes are the same, but modes 4 and 5 are interchanged. Mode 6, 7 and 8 of Ref. [4] correspond to 8, 6 and 7 of Fig. 5. There is an effect caused by the different aspect ratio and the step discontinuities.

Table 4
Comparison between MIN6 and Gorman and Singhal [1] non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a cantilever Mindlin isotropic plate of aspect ratio 3/4 with three-step discontinuities in thickness

Mode number	Type	$L/t = 24$	
		MIN6	Gorman and Singhal [1]
1	Sym	2.573	2.572
2	Antisym	4.858	4.877
3	Sym	7.734	7.732
4	Antisym	10.69	10.72
5	Sym	11.12	11.14
6	Sym	17.57	17.58

Table 5
Comparison between finite element MIN6s results and Gorman and Singhal [1] theoretical and experimentally measured free vibration frequencies for a plate with three-step discontinuities in thickness

Mode number	Type	MIN6 frequency (Hz)	Gorman and Singhal [1]	
			Theoretical frequency (Hz)	Experimental frequency (Hz)
1	Sym	86.746	85.80	86.7
2	Antisym	164.17	167.0	166.0
3	Sym	260.76	278.2	269.0
4	Antisym	360.76	385.4	374.0
5	Sym	375.62	399.6	393.7
6	Sym	591.29	633.2	611.2

Table 6
Non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a cantilever Mindlin isotropic plate of aspect ratio 3 with different span-thickness ratios (Fig. 1)

Mode number	MIN6			
	$L/t = 30$	$L/t = 24$	$L/t = 10$	$L/t = 5$
1	31.239	31.073	29.225	24.774
2	38.691	38.343	34.980	28.407
3	59.089	58.257	50.940	39.143
4	93.548	91.881	78.037	57.666
5	145.15	141.98	116.45	80.343
6	187.42	181.89	137.23	85.763
7	193.88	187.96	141.33	87.267
8	218.79	212.72	160.53	100.00
9	225.21	217.41	166.41	109.27
10	270.88	260.23	188.69	118.73
11	304.43	294.07	218.16	135.10
12	330.59	316.11	226.69	141.41
13	399.53	379.76	262.70	157.36
14	418.36	399.44	280.90	159.31
15	487.02	460.38	306.31	167.07
16	515.65	483.07	311.47	170.57

Table 7
Non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a cantilever Mindlin isotropic plate of aspect ratio 3/2 with one-step discontinuities in thickness and with different span-thickness ratios (Fig. 4)

Mode number	MIN6			
	$L/t = 30$	$L/t = 24$	$L/t = 10$	$L/t = 5$
1	9.2946	9.2809	9.1327	8.6970
2	13.045	12.998	12.565	11.597
3	23.660	23.530	22.389	20.084
4	33.086	32.943	31.314	27.165
5	38.727	38.478	35.956	30.400
6	43.307	43.057	40.744	35.788
7	56.776	56.247	51.185	41.510
8	71.569	71.066	66.379	56.342
9	85.730	84.703	75.338	58.789
10	96.946	95.888	84.807	63.657
11	101.22	100.06	88.344	66.487
12	111.23	110.19	100.22	74.541
13	117.13	115.50	100.55	81.608
14	129.59	127.76	111.39	83.924
15	146.16	143.78	122.72	90.236
16	158.33	156.58	139.32	98.721

Table 8
Non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a square cantilever Mindlin isotropic plate with two-step discontinuities in thickness and with different span-thickness ratios (Fig. 2)

Mode number	MIN6			
	$L/t = 30$	$L/t = 24$	$L/t = 10$	$L/t = 5$
1	4.1331	4.1293	4.0917	3.9860
2	7.5658	7.5450	7.3549	6.9321
3	16.472	16.426	15.946	14.678
4	18.695	18.638	18.074	16.710
5	21.809	21.722	20.828	18.644
6	36.669	36.452	34.324	29.709
7	38.406	38.246	36.655	32.827
8	41.811	41.603	39.251	33.470
9	47.863	47.561	44.355	37.200
10	63.315	62.852	58.243	48.178
11	65.734	65.171	59.520	48.495
12	68.572	68.154	64.062	54.858
13	86.219	85.382	76.588	59.315
14	90.479	89.505	79.717	61.399
15	96.432	95.348	85.012	66.194
16	102.26	101.27	91.307	71.073

Table 9
Non-dimensional frequency $\lambda = \omega L^2(\rho t/D)^{1/2}$ for a cantilever Mindlin isotropic plate of aspect ratio 3/4 with three-step discontinuities in thickness and with different span-thickness ratios (Fig. 3)

Mode number	MIN6			
	$L/t = 30$	$L/t = 24$	$L/t = 10$	$L/t = 5$
1	2.5745	2.5729	2.5584	2.5203
2	4.8667	4.8580	4.7792	4.6094
3	7.7420	7.7345	7.6526	7.4062
4	10.707	10.685	10.463	9.9106
5	11.142	11.121	10.927	10.499
6	17.574	17.540	17.148	16.004
7	20.608	20.550	19.985	18.622
8	21.118	21.065	20.515	18.947
9	21.939	21.883	21.334	20.159
10	34.228	34.073	32.506	28.862
11	35.385	35.266	33.904	30.033
12	36.633	36.522	35.480	32.706
13	38.214	38.081	36.677	33.245
14	40.516	40.336	38.424	34.214
15	53.695	53.388	50.207	43.006
16	55.452	55.249	53.148	46.152

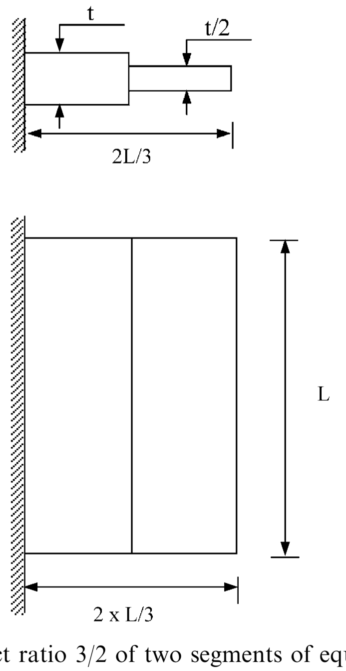

Fig. 4. Cantilever plate with aspect ratio 3/2 of two segments of equal width with different thickness.

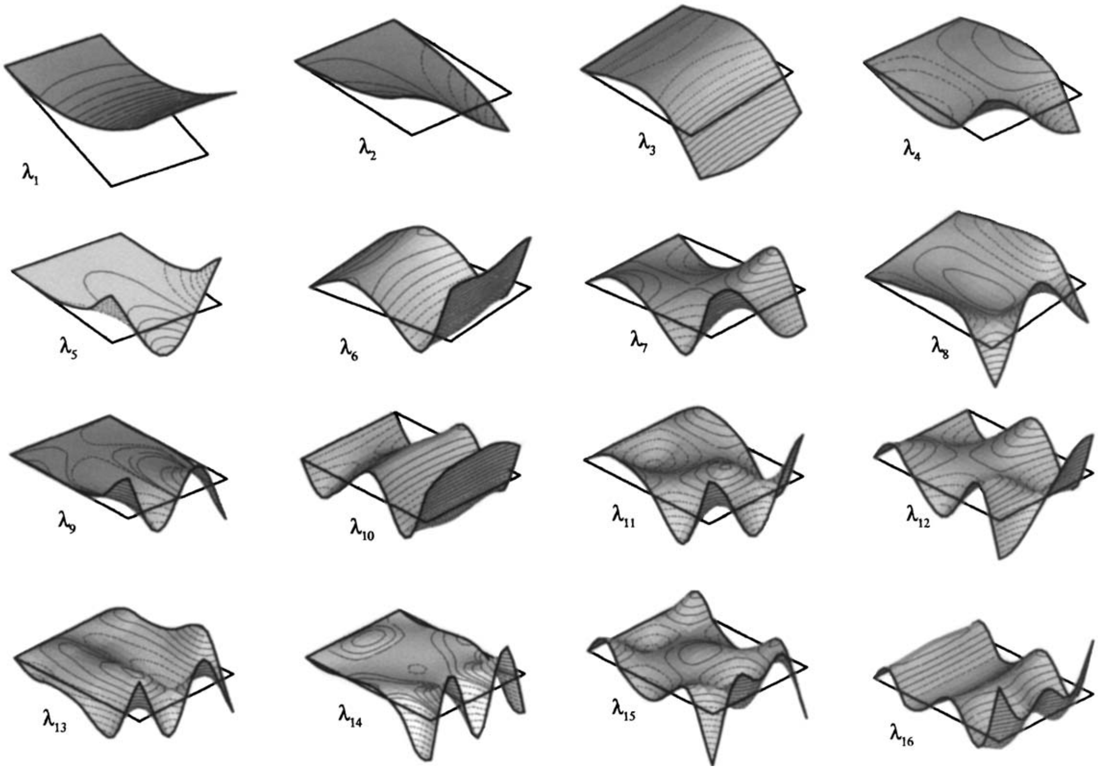

Fig. 5. Mode shapes for a cantilever plate with aspect ratio 3/4 of three step discontinuities and $L/t = 24$ (see Table 9).

4. Conclusions

A new high order, six-node triangular, anisoparametric Mindlin plate finite element has been employed to study the vibration of stepped thickness cantilevered plates. Vibration results were compared and verified with a uniform thickness Mindlin plate analysis and a stepped thickness thin plate analysis. New results were tabulated for a variety of moderately thick cantilevered plates with different step discontinuities. Typical mode shapes were depicted as three-dimensional contour plats.

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